The Cirac-Zoller Controlled-NOT Gate

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Introduction

A quantum computer obeys the laws of quantum mechanics, and its unique feature is that it can follow a superposition of computation paths simultaneously and produce a final state depending on the interference of these paths.

Algorithms exist that are believed to be able to solve some problems efficiently that are considered to be intractible on classical Turing machines, an example being the factorization of large composite numbers into primes, a problem which is the basis of the security of many classical cryptosystems.

The task of designing a quantum computer is equivalent to finding a physical implementation of quantum gates between quantum bits (or qubits), where a qubit refers to a two-state system $\{|0\rangle, |1\rangle\}$.

Introduction

It has been shown that any operation can be decomposed into controlled-NOT gates between two qubits and rotations on a single qubit, where a controlled-NOT gate is defined by

$$\hat{C}_{12} : |\epsilon_1\rangle|\epsilon_2\rangle \rightarrow |\epsilon_1\rangle|\epsilon_1 \oplus \epsilon_2\rangle$$

with $\epsilon_{1,2} = 0, 1$ and $\oplus$ denoting addition modulo 2.

The main obstacle for a practical realization of a quantum computer is the existence of decoherence processes due to the interaction of the system with the environment.

In Phys. Rev. Lett. **74**, 4091 (1995), Cirac and Zoller showed how a set of N cold ions interacting with laser light and moving in a linear trap provides a realistic physical system to implement a quantum computer, and this talk is based on that paper.

The Ion Trap

The distinctive features of the ion trap system are

1. It allows the implementation of n-bit quantum gates between any set of (not necessarily neighbouring) ions
2. Decoherence effects can be made negligible during the whole computation
3. The final readout can be performed with unit efficiency

The basic elements of the computer (i.e. the qubits) are the ions themselves.

The Ion Trap

The two states of the nth ubit are identified with two of the internal states of the corresponding ion; for example, a ground state $|g\rangle_n \equiv |0\rangle_n$ and and excited state $|e\rangle_n \equiv |1\rangle_n$. The state of the quantum computer is a superposition

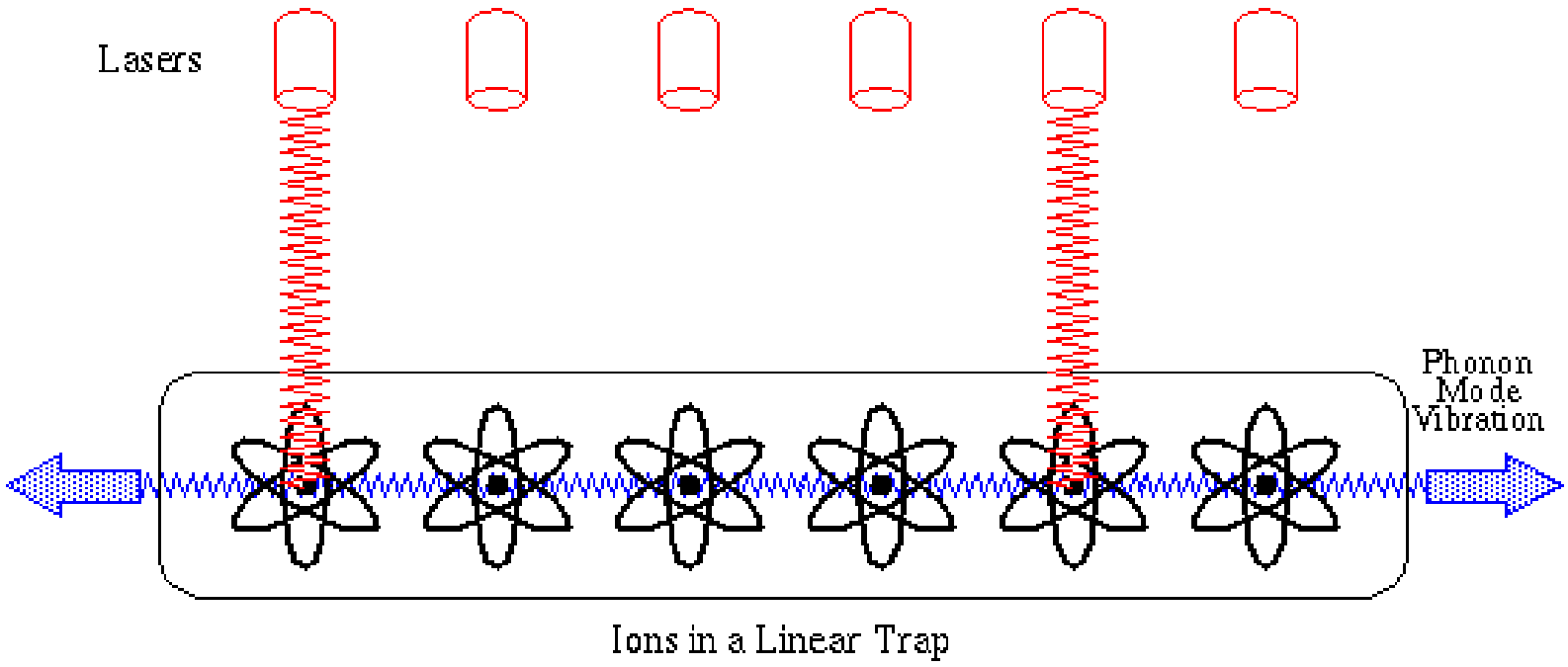
$$|\psi\rangle = \sum_{x=0}^{2^n-1} c_x |x\rangle \equiv \sum_{\underline{x}=\{0,1\}^N} c_{\underline{x}} |\underline{x}\rangle$$

of quantum registers $|\underline{x}\rangle = |x_{N-1}\rangle_{N-1} \cdots |x_0\rangle_0$ with $x = \sum_{n=0}^{N-1} x_n 2^n$ the binary decomposition of x.

The Ion Trap

- Independent manipulation of each qubit is accomplished by directing different laser beams to each of the ions
- The controlled-NOT gate can be implemented by exciting the collective quantized motion of the ions with lasers
- The coupling of the motion of the ions is by the Coulomb repulsion, which is much stronger than any other interaction for typical separations of the ions of a few optical wavelengths
- Decoherence is due to spontaneous decay of the internal atomic states, and damping of the motion of the ions

The Ion Trap



The Ion Trap

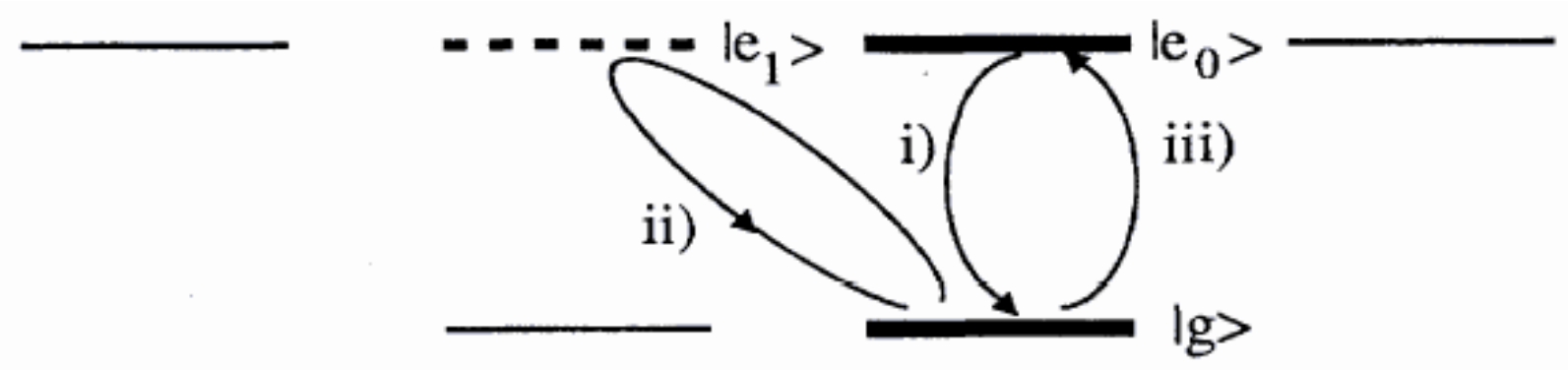
- The confinement of the motion along X, Y and Z directions can be described by an (anisotropic) harmonic potential of frequencies $\nu_x \ll \nu_y, \nu_z$
- The ions are laser cooled so they undergo very small oscillations around the equilibrium position
- In this case the motion of the ions is described in terms of normal modes, and we assume that sideband cooling has left all the normal modes in their ground states
- For this to be possible, the Lamb-Dicke limit (LDL) must hold, implying that the frequencies of the modes must be larger than the photon recoil frequency corresponding to the transition used for laser cooling

The Ion Trap

- For example, for the $S_{1/2} \rightarrow D_{5/2}$ transition of a barium ion, this requires $\nu_{x,y,z} \gg 3$ kHz
- In typical situations $\nu_{y,z} \gg 2\pi\times$ 50 kHz
- The minimum frequency is that of the centre-of-mass (CM) mode moving in the X direction, and coincides with ν_x
- The next frequency is $\sqrt{3}\,\nu_x$ and all the others are larger
- A remarkable feature of this system is that the frequency spacing is independent of the number of ions in the trap

The Ion Trap

Here is a typical atomic level scheme for an alkaline earth ion, corresponding to an electric dipole-forbidden transition:



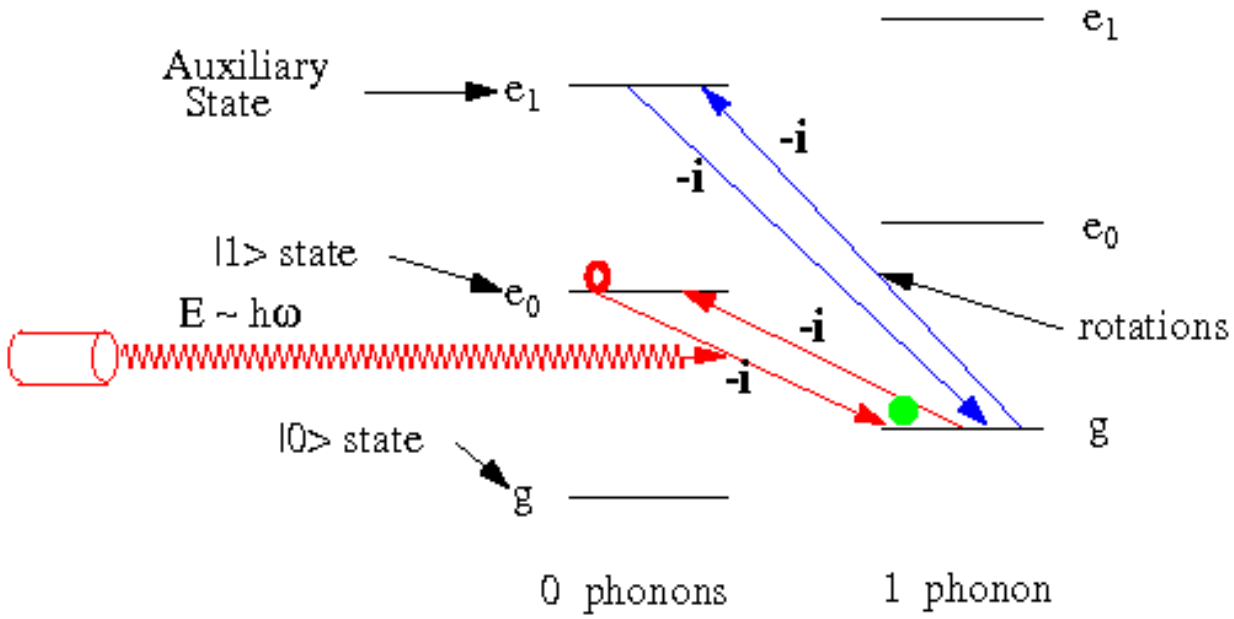
The two-level system chosen as the qubit is marked with thicker lines ($|g\rangle$ and $|e_0\rangle$)

Interactions Between Ions

- When a laser beam acts on one of the ions, it induces transitions between its (internal) ground state and excited levels, and can change the state of the collective normal modes
- For example, with a laser frequency so that the detuning equals minus the CM mode frequency ($\delta_n = -\nu_x$), the CM mode is excited exclusively, because the frequencies of the different normal modes are well separated
- This allows the interactions between the ions to be controlled by appropriately selecting the frequency of the lasers

Interactions Between Ions

The detuned laser stimulates transitions between the $|g\rangle$ state with one phonon and the $|e_0\rangle$ state with no phonons:



Note that the $|e_1\rangle$ state in this diagram does not correspond to the auxiliary state in the earlier diagram, which is the one we will use.

Interactions Between Ions

Consider a laser acting on the nth ion, with the laser frequency chosen such that $\delta_n = -\nu_x$, and the equilibrium position of the ion coinciding with the node of the laser standing wave. The Hamiltonian describing this situation is

$$\hat{H}_{n,q} = \frac{\eta}{\sqrt{N}} \frac{\Omega}{2} [|e_q\rangle_n \langle g| a e^{-i\phi} + |g\rangle_n \langle e_q| a^\dagger e^{i\phi}]$$

where $a^\dagger$ and a are the creation and annihilation operators of the CM phonons respectively, Ω is the Rabi frequency, ϕ is the laser phase, and $\eta = [\hbar k_\theta^2/(2M\nu_x)]^{1/2}$ is the LDL parameter [$k_\theta = k\cos\theta$, with k the laser wave vector, and θ the angle between the X axis and the direction of propagation of the laser]. The subscript $q = 0, 1$ refers to the transition excited by the laser, which depends on the laser polarization (see atomic level diagram).

Interactions Between Ions

Now, recall that the Schrödinger equation

$$i\hbar \frac{d|\psi\rangle}{dt} = \hat{H}|\psi\rangle$$

has the solution

$$\exp\left(\frac{-i\hat{H}t}{\hbar}\right)$$

so if the laser is on for a time $t = k\pi/(\Omega\,\eta/\sqrt{N})$ (i.e. a $k\pi$ pulse), and setting $\hbar = 1$, the evolution of the system will be described by the unitary operator

$$\hat{U}_n^{k,q} = \exp(-ik\pi/(\Omega\,\eta/\sqrt{N})\hat{H}_{n,q})$$

Interactions Between Ions

$$\hat{U}_n^{k,q}(\phi) = \exp\left[-ik\frac{\pi}{2}(|e_q\rangle_n\langle g|ae^{-i\phi} + |g\rangle_n\langle e_q|a^\dagger e^{i\phi})\right]$$

We can express this operator exponential as a power series. Let

$$\hat{A} = |e_q\rangle_n\langle g|ae^{-i\phi} + |g\rangle_n\langle e_q|a^\dagger e^{i\phi}$$

and begin by evaluating the quadratic term

$$\begin{aligned}\hat{A}^2 &= |e_q\rangle_n\langle g|e_q\rangle_n\langle g|a^2e^{-2i\phi} + |e_q\rangle_n\langle g|g\rangle_n\langle e_q|aa^\dagger + \\ &\quad |g\rangle_n\langle e_q|e_q\rangle_n\langle g|a^\dagger a + |g\rangle_n\langle e_q|g\rangle_n\langle e_q|(a^\dagger)^2e^{2i\phi} \\ &= |e_q\rangle_n\langle e_q|aa^\dagger + |g\rangle_n\langle g|a^\dagger a\end{aligned}$$

Interactions Between Ions

$$\hat{A}^2 = |e_q\rangle_n\langle e_q|aa^\dagger + |g\rangle_n\langle g|a^\dagger a$$

Now, let $|0\rangle$ and $|1\rangle$ denote the states of the CM mode with zero and one phonons respectively, and recall that $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. Therefore

$$\begin{aligned} a|1\rangle &= |0\rangle \\ a^\dagger|0\rangle &= |1\rangle \end{aligned}$$

and

$$\hat{A}^2|g\rangle_n|1\rangle = |g\rangle_n|1\rangle$$

$$\hat{A}^2|e_q\rangle_n|0\rangle = |e_q\rangle_n|0\rangle$$

so for these two particular states $\hat{A}^2$ performs the identity operation

Interactions Between Ions

For the the exponential function we have

$$e^{\hat{A}} = I + \hat{A} + \frac{\hat{A}^2}{2!} + \frac{\hat{A}^3}{3!} + \frac{\hat{A}^4}{4!} + \frac{\hat{A}^5}{5!} + \cdots$$

so

$$e^{i\theta\hat{A}} = I + i\theta\hat{A} - \frac{(\theta\hat{A})^2}{2!} - i\frac{(\theta\hat{A})^3}{3!} + \frac{(\theta\hat{A})^4}{4!} + i\frac{(\theta\hat{A})^5}{5!} + \cdots$$

and in the special case that $\hat{A}^2 = I$

$$e^{i\theta\hat{A}} = I + i\theta\hat{A} - \frac{\theta^2 I}{2!} - i\frac{\theta^3\hat{A}}{3!} + \frac{\theta^4 I}{4!} + i\frac{\theta^5\hat{A}}{5!} + \cdots$$

Interactions Between Ions

$$e^{i\theta\hat{A}} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) I + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)\hat{A}$$

$$e^{i\theta\hat{A}} = \cos(\theta) I + i\sin(\theta)\hat{A}$$

and therefore when acting on the states $|g\rangle_n|1\rangle$ and $|e_q\rangle_n|0\rangle$ we can express the exponential

$$\hat{U}_n^{k,q}(\phi) = \exp\left[-ik\frac{\pi}{2}(|e_q\rangle_n\langle g|ae^{-i\phi} + |g\rangle_n\langle e_q|a^\dagger e^{i\phi})\right]$$

as

$$\hat{U}_n^{k,q}(\phi) = \cos(k\pi/2)I - i\sin(k\pi/2)\left[|e_q\rangle_n\langle g|ae^{-i\phi} + |g\rangle_n\langle e_q|a^\dagger e^{i\phi}\right]$$

Interactions Between Ions

$$\hat{U}_n^{k,q}(\phi) = \cos(k\pi/2)I - i\sin(k\pi/2)\left[|e_q\rangle_n\langle g|ae^{-i\phi} + |g\rangle_n\langle e_q|a^\dagger e^{i\phi}\right]$$

so

$$\hat{U}_n^{k,q}(\phi)|g\rangle_n|1\rangle = \cos(k\pi/2)|g\rangle_n|1\rangle - ie^{-i\phi}\sin(k\pi/2)|e_q\rangle_n|0\rangle$$

$$\hat{U}_n^{k,q}(\phi)|e_q\rangle_n|0\rangle = \cos(k\pi/2)|e_q\rangle_n|0\rangle - ie^{i\phi}\sin(k\pi/2)|g\rangle_n|1\rangle$$

Furthermore, acting on the state $|g\rangle_n|0\rangle$ we have

$$\begin{aligned}\hat{A}|g\rangle_n|0\rangle &= \left[|e_q\rangle_n\langle g|ae^{-i\phi} + |g\rangle_n\langle e_q|a^\dagger e^{i\phi}\right]|g\rangle_n|0\rangle = 0\\ \hat{A}^2|g\rangle_n|0\rangle &= \left[|e_q\rangle_n\langle e_q|aa^\dagger + |g\rangle_n\langle g|a^\dagger a\right]|g\rangle_n|0\rangle = 0\end{aligned}$$

so the only non-zero term in the exponential power series is I and the state is unchanged by $\hat{U}_n^{k,q}(\phi)$.

The Controlled-NOT Gate

We now have all we need to implement a two-qubit gate. Consider the following three-step process (see atomic level diagram)

1. A π laser pulse with polarization $q = 0$ and $\phi = 0$ excites the mth ion. The evolution corresponding to this step is given by $\hat{U}_m^{1,0}(0)$
2. The laser directed on the nth ion is then turned on for the time of a 2π pulse with polarization $q = 1$ and $\phi = 0$. The corresponding evolution operator $\hat{U}_n^{2,1}$ changes the sign of the state $|g\rangle_n|1\rangle$ (without affecting the others) via a rotation through the auxiliary state $|e_1\rangle_n|0\rangle$
3. Same as the first step

The unitary operation for the whole process is $\hat{U}_{m,n} = \hat{U}_m^{1,0}\hat{U}_n^{2,1}\hat{U}_m^{1,0}$

The Controlled-NOT Gate

$$
\begin{array}{ccccc}
 & \hat{U}_m^{1,0} & & \hat{U}_n^{2,1} & \\
|g\rangle_m|g\rangle_n|0\rangle & \rightarrow & |g\rangle_m|g\rangle_n|0\rangle & \rightarrow & |g\rangle_m|g\rangle_n|0\rangle \\
|g\rangle_m|e_0\rangle_n|0\rangle & \rightarrow & |g\rangle_m|e_0\rangle_n|0\rangle & \rightarrow & |g\rangle_m|e_0\rangle_n|0\rangle \\
|e_0\rangle_m|g\rangle_n|0\rangle & \rightarrow & -i|g\rangle_m|g\rangle_n|1\rangle & \rightarrow & i|g\rangle_m|g\rangle_n|1\rangle \\
|e_0\rangle_m|e_0\rangle_n|0\rangle & \rightarrow & -i|g\rangle_m|e_0\rangle_n|1\rangle & \rightarrow & -i|g\rangle_m|e_0\rangle_n|1\rangle
\end{array}
$$

$$
\begin{array}{cc}
\hat{U}_m^{1,0} & \\
|g\rangle_m|g\rangle_n|0\rangle & \rightarrow \quad |g\rangle_m|g\rangle_n|0\rangle \\
|g\rangle_m|e_0\rangle_n|0\rangle & \rightarrow \quad |g\rangle_m|e_0\rangle_n|0\rangle \\
i|g\rangle_m|g\rangle_n|1\rangle & \rightarrow \quad |e_0\rangle_m|g\rangle_n|0\rangle \\
-i|g\rangle_m|e_0\rangle_n|1\rangle & \rightarrow \quad -|e_0\rangle_m|e_0\rangle_n|0\rangle
\end{array}
$$

The Controlled-NOT Gate

- The effect of this interaction is to change the sign of the state only when both ions are initially excited
- Note that the state of the CM mode is restored to the vacuum state $|0\rangle$ after the process
- This operation is equivalent to a controlled-NOT gate

The Controlled-NOT Gate

To see this, let

$$|\pm\rangle = (|g\rangle \pm |e_0\rangle)/\sqrt{2}$$

The operation can then be summarized as

$$|g\rangle_m|\pm\rangle_n \rightarrow |g\rangle_m|\pm\rangle_n$$

$$|e_0\rangle_m|\pm\rangle_n \rightarrow |e_0\rangle_m|\mp\rangle_n$$

With an appropriate one-bit rotation applied to the nth ion this procedure then yields the controlled-NOT. These rotations acting on a single ion can be performed using a laser frequency on resonance with the internal transition ($\delta_n = 0$), polarization $q = 0$, and with the equilibrium position of the ion coinciding with the antinode of the laser standing wave.

The Controlled-NOT Gate

The Hamiltonian is

$$\hat{H}_n = (\Omega/2)[|e_0\rangle_n\langle g|e^{-i\phi} + |g\rangle_n\langle e_0|e^{i\phi}]$$

and for an interaction time $t = k\pi/\Omega$ (i.e. using a $k\pi$ pulse), this process is described by the unitary evolution operator:

$$V_n^k(\phi) = \exp\left[-ik\frac{\pi}{2}(|e_0\rangle_n\langle g|e^{-i\phi} + |g\rangle_n\langle e_0|e^{i\phi})\right]$$

so that

$$|g\rangle_n \rightarrow \cos(k\pi/2)|g\rangle_n - ie^{-i\phi}\sin(k\pi/2)|e_0\rangle_n$$

$$|e_0\rangle_n \rightarrow \cos(k\pi/2)|e_0\rangle_n - ie^{i\phi}\sin(k\pi/2)|g\rangle_n$$

The Controlled-NOT Gate

Thus the complete controlled-NOT gate for the states $|\epsilon_m\rangle|\epsilon_n\rangle$ $(\epsilon_{m,n} = g, e_0)$ is given by

$$\hat{C}_{mn} = \hat{V}_n^{1/2}(\frac{\pi}{2})\hat{U}_{m,n}\hat{V}_n^{1/2}(-\frac{\pi}{2})$$

The (controlled)^n-NOT gate between n arbitrary ions in the trap can be implemented in a similar way, and although such a gate can be decomposed into a finite number of controlled-NOT gates plus one-qubit rotations, this may require many operations.

Summary

The two key elements behind the implementation of the controlled-NOT gate are

- Non-local entanglement between individual qubits, achieved by transferring the internal atomic coherence to and from the CM motion shared by all the ions ($\hat{U}_n^{k=1,q=0}$)
- An intermediate step where we 'hide the atomic amplitudes' in a third internal atomic level $|e_1\rangle$ ($\hat{U}_n^{k=1,q=1}$) (for three-qubit gates) and induce 2π rotations via this state to selectively change the sign of atomic amplitudes $\hat{U}_n^{k=2,q=1}$

Note that no population is left in the auxiliary atomic and CM levels after the complete gate operation, so any population left in these states could be used to implement an error detection scheme.