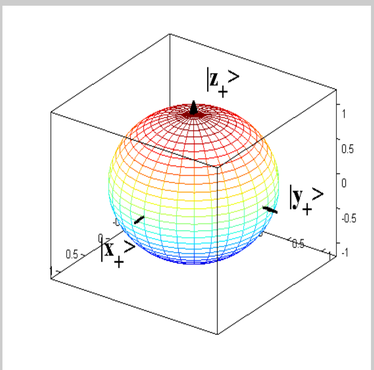
Conjugate Variables as a Resource in Signal and Image Processing

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INTRODUCTION

Applying the formalism of quantum information theory to statistics and probability theory [1, 2] and signal and image processing applications [3, 4] can result in novel classical algorithms, which we call Quantum Inspired Algorithms (QIA) in analogy to the term Quantum Signal Processing coined by Eldar [4]. A crucial factor seems to be the ability to represent relations between two (or more) sources of information jointly as quantum information. In analogy to the behaviour of physical conjugate variables, we define *conjugate information variables*, with the property that whenever one of them is highly predictable the other one is unpredictable and vice versa. As an example, consider a classification task to decide whether a function is locally constant, rising, falling or minimum or maximum, in the presence of noise.



ENCODING A SIGNAL AS QUANTUM INFORMATION

Given a signal $f(t)$, we use the derivatives $f'(t)$ and $f''(t)$ as conjugate variables. We let

$$p^{\pm}(t) = \sqrt{\frac{c_2 \pm f''(t)}{2c_2}}$$
$$\alpha^{+}(t) = p^{+}(t)e^{i\pi f'(t)/c_1}$$
$$\alpha^{-}(t) = p^{-}(t)$$
$$\text{with } |f'(t)| \leq c_1, |f''(t)| \leq c_2 \text{ for all } t$$

and define quantum states

$$|\psi(t)\rangle = \alpha^{+}(t)\,|z_+\rangle + \alpha^{-}(t)\,|z_-\rangle$$

Note that as $|\alpha^{+}(t)|^2 + |\alpha^{-}(t)|^2 = 1$, for all t, the $|\psi(t)\rangle$ are well-defined qubits. $|z_+\rangle, |z_-\rangle$ are the $+z$ and $-z$ axes of the Bloch sphere.

CONJUGATE VARIABLES

$f''(t) = \pm c_2 \iff |\psi(t)\rangle = |z_\pm\rangle$ independent of $f'(t)$. The positive and negative x and y axes are given by

$$|x_+\rangle = \tfrac{1}{\sqrt{2}}(|z_+\rangle + |z_-\rangle), \quad |x_-\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle - |z_-\rangle)$$
$$|y_+\rangle = \tfrac{1}{\sqrt{2}}(|z_+\rangle + i\,|z_-\rangle), \quad |y_-\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle - i\,|z_-\rangle)$$

$f''(t) = f'(t) = 0 \Leftrightarrow |\psi(t)\rangle = |x_+\rangle$, $f''(t) = 0 \wedge f'(t) = \pm c_1 \Leftrightarrow |\psi(t)\rangle = |x_-\rangle$, so $f'(t)$ and $f''(t)$ are conjugate variables in the sense that, given the random variables $X = \{|x_+\rangle\langle x_+|, |x_-\rangle\langle x_-|\}$ and $Z = \{|z_+\rangle\langle z_+|, |z_-\rangle\langle z_-|\}$, the commutator between any two projectors of X and Z is maximal:

$$A_\pm = |x_\pm\rangle\langle x_\pm|, B_\pm = |z_\pm\rangle\langle z_\pm| \Rightarrow \|A_\pm B_\pm - B_\pm A_\pm\|^2 = \frac{1}{2}$$

PROJECTORS

Next we define a set of projectors:

$$P_0 = |z_+\rangle\langle z_+|$$
$$P_1 = |z_-\rangle\langle z_-|$$
$$P_2 = (|y_-\rangle + |x_-\rangle)(\langle y_-| + \langle x_-|)/2$$
$$P_3 = (|y_+\rangle + |x_-\rangle)(\langle y_+| + \langle x_-|)/2$$
$$P_4 = |x_+\rangle\langle x_+|$$

Only P_0 and P_1 form a random variable as they add up to the identity and therefore fulfil the completeness condition. They indicate minimum or maximum of the function. $P_i, i = 2, \ldots, 4$ would have to be complemented by $\mathbb{I} - P_i$ in order to do so. The choice of projectors depends on the problem we want to solve.

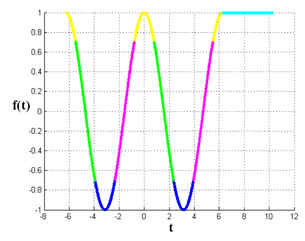
CLASSIFICATION

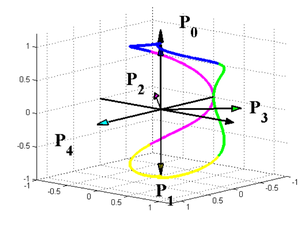
For any given $f(t)$ we now can project the corresponding $|\psi(t)\rangle$ and get a classification result by maximizing over all projectors

$$C(f(t)) = i \in \{0, \ldots, 4\} : \langle\psi(t)|\,P_i\,|\psi(t)\rangle = \max_{j=0,\ldots,4} \langle\psi(t)|\,P_j\,|\psi(t)\rangle$$

The colour-coded classification for the analytic function $f(t)$ is shown below:

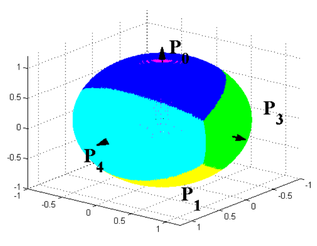
$$f(t) = \begin{cases} \cos(t), & |t| < 2\pi \\ 1, & t > 2\pi \end{cases}$$

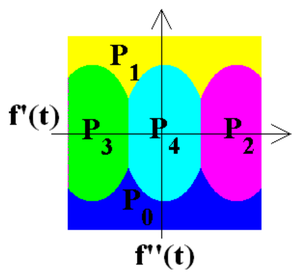


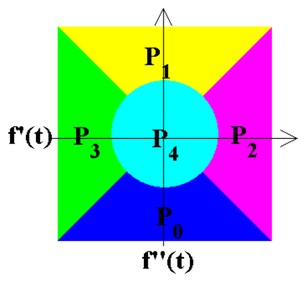


COMPARISON WITH A STANDARD MODEL-BASED APPROACH

Left: Decision regions on the Bloch sphere. **Middle:** Decision regions over the joint distribution of $f''(t)$ and $f'(t)$. **Right:** Model-based decision regions. The limits of the axes are $\pm c_1$ for $f'(t)$ and $\pm c_2$ for $f''(t)$. If $c = c_1 = c_2$, the radius of the inner circle, P_4, is given by $c/2$.



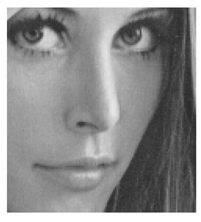




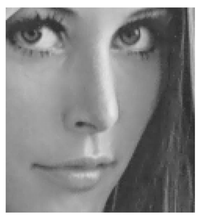
The model-based decision depends on both signals over almost the entire joint distribution, but the decision based on conjugate variables is insensitive to one signal around the poles of the Bloch sphere.

APPLICATION TO ADAPTIVE IMAGE FILTERING

The original image (left) was smoothed on the constant part P_4 of the image function as given in the picture at the right. The result of smoothing adaptively with an average filter (15×15) is shown in the middle-right picture ($P_0, \ldots, P_3$ are not filtered). This results in an image that is smooth in these constant image areas but untouched elsewhere.









The middle-left picture shows the result using the same filter kernel independent of the image content.

REFERENCES

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[4] Yonina C. Eldar and Alan V. Oppenheim, *Quantum signal processing*, IEEE Signal Processing Magazine (2002).